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FAST TRACK COMMUNICATION

Discretely holomorphic parafermions and integrable loop models

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Abstract
We define parafermionic observables in various lattice loop models, including examples where no Kramers–Wannier duality holds. For a particular rhombic embedding of the lattice in the plane and a value of the parafermionic spin these variables are discretely holomorphic (they satisfy a lattice version of the Cauchy–Riemann equations) as long as the Boltzmann weights satisfy certain linear constraints. In the cases considered, the weights then also satisfy the critical Yang–Baxter equations, with the spectral parameter being related linearly to the angle of the elementary rhombus.

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1. Introduction

Holomorphic (and antiholomorphic) fields are the basic building blocks of conformal field theories (CFTs). They have simple short-distance expansions among themselves and with other fields of the theory. Although the prototype of such holomorphic field is the stress tensor $T(z)$, which is present in all CFTs, many interesting CFTs contain holomorphic fields with fractional conformal spin. These are often referred to as parafermionic. The correlation functions of such holomorphic fields are necessarily power behaved, and therefore, in local theories, can exist only in the massless, conformal case.

Many CFTs are also believed to describe the scaling limit of critical lattice models. In order to understand the emergence of holomorphic fields in this limit, the simplest possibility is that there should exist analogues already in the discrete setting. These we refer to as *discretely holomorphic observables* of the lattice model. A precise definition is given below.

Recently a number of examples of such observables have been discovered in well-known lattice models. In models which possess a Kramers–Wannier duality symmetry, parafermionic observables can be defined in terms of suitable products of neighbouring order and disorder variables [1]. In the case of the $\mathbb{Z}_N$, or clock, models, it was shown in [2] that such parafermions are indeed discretely holomorphic, but only at the *integrable* critical points identified by Fateev and Zamolodchikov [3]. In this case, the parafermions were defined algebraically directly in terms of the spin variables and Boltzmann weights of the model.

A second example is the Q-state Potts model, which also possesses a duality symmetry. This model, in its Fortuin–Kasteleyn (FK) random cluster form, can also be represented in terms of a set of dense random curves [4]. In [5], it was shown that the parafermions defined in terms of order–disorder products have support on these curves, and indeed may be considered as observables depending only on the curves. For example, the two-point function vanishes unless the two points happen to lie on the same curve. In [5, 6] a simple argument was given that these parafermionic observables are discretely holomorphic, but only at the critical point of the Potts model (where it is also known to be integrable).

In this communication, we amplify these comments and enlarge the list of lattice models for which discretely holomorphic observables can be identified. In particular, we consider various versions of the O(n) model which do not possess Kramers–Wannier duality and for which the definition of a parafermion in terms of an order–disorder product fails. However, these models all have a representation in terms of curves on the lattice, and using this we show that one may still define parafermionic observables, generalizing the construction in the Potts model. Moreover, in all cases, these turn out to be discretely holomorphic precisely when the model is integrable and critical.

More generally, we may consider versions of these models where the weights are anisotropic. In this case, one would expect the scaling limit to be a rotationally invariant CFT only if the lattice is embedded in the plane in a particular way. In general, there is a one-parameter family of such inequivalent embeddings of a homogeneous lattice. For a given embedding, we find that discrete holomorphicity holds only for a particular set of weights. These weights, in all cases, satisfy the Yang–Baxter equation, with the usual spectral parameter being simply related to the angle of shear in the embedding.

This is a remarkable result, since the requirement of discrete holomorphicity, as we will show, gives a set of linear equations in the weights for a fixed value of the spectral parameter. On the other hand, the Yang–Baxter equations are cubic functional relations between the weights at different values of the spectral parameter.

Let us define more carefully the notion of discrete holomorphicity. Let $\mathcal{G}$ be a planar graph (which will usually be a regular lattice) embedded in the complex plane, with vertices at points $\{z_j\}$. Let $F(z)$ be a function defined on the vertices of the medial lattice: the midpoints $\frac{1}{2}(z_i + z_j)$ of the edges (ij) of $\mathcal{G}$. Then F is discretely holomorphic on $\mathcal{G}$ if

$$\sum_{(ij)\in\mathcal{P}} F\left((z_i + z_j)/2\right)(z_j - z_i) = 0, \qquad (1)$$

where the sum is over the edges of each face, or plaquette, $\mathcal{P}$ of $\mathcal{G}$. Equation (1) may be thought of as a discrete version of Cauchy's theorem. However, note that, since there are fewer equations than unknowns, even in the discrete setting (1) is not sufficient to solve a boundary value problem on $\mathcal{G}$. (This should be compared with the example of the discrete Laplace equation.) In addition, if we try to take the scaling limit by covering the interior of some domain $\mathcal{D}$ in the plane by a suitable sequence of graphs $\mathcal{G}$ whose mesh size approaches zero, we may deduce that contour integrals vanish, but Morera's theorem implies that the scaling limit of F is analytic only if we first assume, or prove, that it is continuous. (In the Ising case, where the scaling limit has been proven [7], it was in fact necessary to adopt another less direct approach.)

In the examples we discuss, the discrete holomorphicity follows entirely from local properties of the lattice model, and therefore we may in fact take $F(z)$ to be any correlation function of the local parafermionic observable with other local fields at other locations, which may be thought of as a conditional expectation value. In what follows we shall not distinguish explicitly between the observable and its correlation functions.

We note that Smirnov [6] has argued that if one can find suitable discretely holomorphic parafermionic observables of curves in lattice models, and also prove that they have a holomorphic scaling limit satisfying suitable boundary conditions, such that their dependence on z in simple domains is computable, then it follows that the scaling limit of the curves is Schramm–Loewner evolution (SLE_κ) with a value of κ related to the conformal spin. This programme has been carried to completion for the Ising model, for the curves forming boundaries of both the spin clusters and the FK clusters [7].

This application of discrete holomorphicity is not the purpose of the present communication. Rather it is to point out that it holds in a wider class of models than was observed up to now, and that it appears to be intimately related to integrability.

However, we should note a general feature of the relation of our results to SLE. As was first observed in [8, 9], the statement that a curve starting on the boundary of a domain is SLE is equivalent in CFT to the assertion that the boundary operator which inserts the curve corresponds to a Virasoro highest weight representation with a null state at level 2, usually denoted by $\phi_{2,1}$ (or $\phi_{1,2}$). It has conformal weight $h_{2,1} = (6-\kappa)/(2\kappa)$. Now one of the important properties of a *holomorphic* bulk conformal field $\psi_s(z)$ with spin s is that, when taken to the boundary, it gives a boundary field with the *same* total conformal weight, that is s. If the bulk field is an observable which depends on N curves, which meet locally at the point z but begin at different points $\{x_j\}$ on the boundary, this implies that the CFT correlation function

$$\left\langle \psi_s(z) \prod_{j=1}^{N} \phi_{2,1}(x_j) \right\rangle$$

is non-zero. By the fusion rules of boundary CFT, this implies that ψ_s must be a $\phi_{N+1,1}$ field, and therefore that

$$s = h_{N+1,1} = \frac{N(2N+4-\kappa)}{2\kappa}.$$

Thus if one knows the value of s one may infer the value of κ. In this communication, we give examples with $N = 1, 2$, as well as cases when the curve cannot be simple SLE.

We note that in this communication we shall consider homogeneous, translationally invariant lattices, but many of the arguments extend to so-called Baxter lattices, which may be embedded in the plane isoradially (that is all their faces are rhombi). These have been discussed with respect to the $\mathbb{Z}_N$ models in [2] and the Potts model in [6]. Discretely holomorphic functions in the Ising model are also studied in [10].

The layout of this communication is as follows. In section 2 we review the arguments for the FK representation of the Potts model. In section 3 we give a parafermionic observable for Nienhuis' O(n) model on the square lattice [11], and show that it is discretely holomorphic precisely on the integrable manifolds [11]. In section 4 we extend this to a model [12] with two different kinds of loops which may cross each other. Our arguments in these last two cases are generalizations of those of Smirnov [13] for the O(n) model on the honeycomb lattice.

2. The self-dual Potts model on the square lattice

The Q-state Potts model is a classical spin model on the lattice. Each vertex carries a spin variable $S_i \in \{1, 2, \ldots, Q\}$, and the Boltzmann weight for a spin configuration is

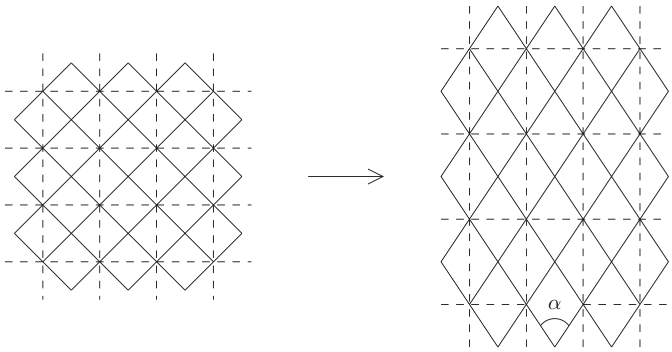

Figure 1. Left: the original square lattice $\mathcal{L}$ of the Potts spins (dotted lines) and the covering lattice $\mathcal{M}^*$ (full lines). Right: the deformed lattices, defined by the angle α.

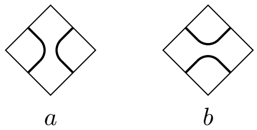

Figure 2. The elementary plaquette configurations for the loop model and their Boltzmann weights. Any closed loop has weight $\sqrt{Q}$.

$$W[\{S_i\}] = \prod_{\langle i,j\rangle} \exp[J_{ij}\delta(S_i, S_j)], \tag{2}$$

where the product is on the edges of the lattice, and J_{ij} is the coupling constant attached to edge $\langle i, j\rangle$. We consider the case of the square lattice $\mathcal{L}$, with coupling constants J_1, J_2 on the horizontal and vertical edges, respectively. The model can be reformulated [4] as a dense loop model on the medial lattice $\mathcal{M}$: this is the lattice consisting of the midpoints of the original lattice. The elementary configurations of the loop model are defined on square plaquettes $\mathcal{P}$, which are the faces of the covering lattice $\mathcal{M}^*$, the dual of $\mathcal{M}$ (see the left-hand side of figure 1). The Boltzmann weights of the elementary plaquettes (see figure 2) are (a_r, b_r), where

$$\frac{b_1}{a_1} = \frac{\mathrm{e}^{J_1} - 1}{\sqrt{Q}}, \qquad \frac{b_2}{a_2} = \frac{\sqrt{Q}}{\mathrm{e}^{J_2} - 1}, \tag{3}$$

and $r = 1, 2$, respectively if the plaquette contains a horizontal or a vertical edge of the original lattice. Each closed loop carries a weight $\sqrt{Q}$. Equivalently, the loop model Boltzmann weights can be encoded in the $\check{R}$-matrix $\check{R}(u) = a(u)1 + b(u)E$, where E (the Temperley–Lieb generator) is the operator acting on loop diagrams as depicted on the right of figure 2, and u is called the spectral parameter.

In this communication, we will restrict ourselves to the self-dual line:

$$(\mathrm{e}^{J_1} - 1)(\mathrm{e}^{J_2} - 1) = Q, \qquad J_1, J_2 > 0, \tag{4}$$

in the case that the loop model is homogeneous. For $0 \leqslant Q \leqslant 4$, the Potts model has a second-order transition on this line. In this regime, we can write

$$\sqrt{Q} = 2\cos\gamma, \qquad 0 \leqslant \gamma \leqslant \frac{\pi}{2}. \tag{5}$$

In [5, 6], a lattice holomorphic observable $F_s(z)$ was identified in this loop formulation. The observable $F_s(z)$ is defined on the midpoints of the plaquette edges as follows:

$$F_s(z) = \sum_{G\in\Gamma(0,z)} P(G)\,\mathrm{e}^{-\mathrm{i}s\theta(z)}, \tag{6}$$

where $P(G)$ is the probability of the graph G, $\Gamma(0,z)$ is the set of loop configurations for which the points 0 and z belong to the same loop, and $\theta(z)$ is the winding angle of this loop from 0 to z. More precisely, we fix arbitrarily a point 0 on the edge of a plaquette (this could be in the bulk or on the boundary of the domain), and a direction for the loop segment passing through 0. The winding angle $\theta(z)$ is defined incrementally, setting $\theta(0) = 0$, and adding $\pm\frac{\pi}{2}$ for each elementary plaquette on the loop, according to the turn made by the loop. The real number s is the spin of the parafermion $F_s(z)$. In [5], the holomorphicity relation (1) was established for F_s defined in (6), at the isotropic self-dual point: $J_1 = J_2 = \ln(1+\sqrt{Q})$. This is only true if the spin s satisfies

$$\sqrt{Q} = 2\sin\frac{\pi s}{2}. \tag{7}$$

In this section, we adapt these results to the whole (anisotropic) self-dual line.

On the one hand, since the loop plaquettes obey a Temperley–Lieb algebra, the $\check{R}$-matrix satisfies the Yang–Baxter equations

$$\check{R}_{12}(u)\check{R}_{23}(v-u)\check{R}_{12}(v) = \check{R}_{23}(v)\check{R}_{12}(v-u)\check{R}_{23}(u), \tag{8}$$

for the parametrization $a(u) = \sin u$, $b(u) = \sin(\gamma - u)$. On the other hand, let us consider the following *deformed model*: this is the self-dual Potts model with (4) and anisotropic weights $J_1 \neq J_2$, defined on the rectangular lattice with vertical edges scaled by a factor $\cot\mathrm{an}\frac{\alpha}{2}$ (see figure 1). Various arguments (see, e.g., [14, 15]) support the statement that, in the continuum limit, the deformed model has the same behaviour as the isotropic self-dual Potts model, if the angle α is related correctly to the ratio J_2/J_1. We shall obtain the following results for the deformed model:

(i) F_s is holomorphic on the lattice if the spin s satisfies the relation (7).
(ii) There exists a linear relation between the spectral parameter u and the angle α.

Let us describe in more details the deformed geometry: the plaquettes become rhombi with internal angles α, $\pi-\alpha$, and the increments of $\theta(r)$ are now $\pm\alpha$, $\pm(\pi-\alpha)$ on each plaquette. The left-hand side of (1) has only contributions from the loop configurations in which a closed loop C connects 0 to two edges of the plaquette $\mathcal{P}$. Let G be one such configuration, where the plaquette $\mathcal{P}$ is in configuration (a) of figure 2, and G' the configuration differing from G only by the configuration of the plaquette $\mathcal{P}$. Our method is to show that the contributions to the left-hand side of (1) from G, G' cancel each other. There are essentially two inequivalent external connectivities of the loop outside of $\mathcal{P}$, as shown in figure 3. In the case (1), the probabilities of G, G' are related by $P(G')/b = P(G)/(a\sqrt{Q})$, whereas in case (2), one has $P(G')/(b\sqrt{Q}) = P(G)/a$. The requirement of cancellation of contributions from G, G' yields a linear system for the Boltzmann weights a, b:

$$\begin{cases}(1+\mu)\sqrt{Q}a + (1+\mu-\lambda-\mu\lambda^{-1})b = 0,\\ (1+\mu-\lambda^{-1}-\mu\lambda^{-1})a + (1-\mu\lambda^{-1})\sqrt{Q}b = 0,\end{cases} \tag{9}$$

where we have set $\lambda = \mathrm{e}^{\mathrm{i}\pi s}$, $\mu = \mathrm{e}^{\mathrm{i}\alpha(s+1)}$. The determinant of the system reads

$$\lambda^{-1}(1+\mu)(1-\mu\lambda^{-1})[\lambda^2 + (Q-2)\lambda + 1]. \tag{10}$$

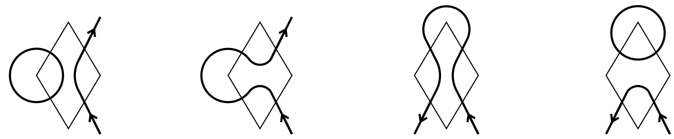

Figure 3. Loop configurations with a loop connecting 0 to the elementary plaquette $\mathcal{P}$.

This has to vanish for the system to have a non-trivial solution. The last factor vanishes for $\lambda = -e^{\pm 2i\gamma}$. We choose the minus sign, corresponding to the exponent

$$s = 1 - \frac{2\gamma}{\pi}. \tag{11}$$

Note that this value satisfies the condition (7). The relative Boltzmann weight is then

$$\frac{b}{a} = -\frac{\cos\frac{(s+1)\alpha}{2}}{\cos\left[\gamma + \frac{(s+1)\alpha}{2}\right]}. \tag{12}$$

Comparing with the parametrization $a(u), b(u)$ for the $\check{R}$-matrix, we get

$$\frac{(s+1)\alpha}{2} = \frac{\pi}{2} - u. \tag{13}$$

Therefore, we recover the result of [15], that the angle α is linearly related to the spectral parameter u. Note that (13) is different from the usual relation $\alpha = \pi u/\gamma$. This can be corrected by setting the loop's angle increments on a plaquette to $\beta, \pi - \beta$. Then the system (9) still holds, if we set $\mu = e^{i(\alpha+\beta s)}$. The angle β can then be tuned to recover the appropriate relation between α and u.

The scaling limit of the self-dual Potts model is described by a Coulomb gas with central charge and conformal weights given by

$$c = 1 - \frac{6(1-g)^2}{g} \qquad h_{r,r'} = \frac{(gr - r')^2 - (1-g)^2}{4g}, \tag{14}$$

where the coupling constant g is given by $g = 1 - \frac{\gamma}{\pi}$. One can then write (11) as $s = h_{3,1}$. According to our earlier observation, this is consistent with the fact that there are *two* curves meeting at the observation point z.

3. The O(n) loop model on the square lattice

In this section, we find solutions of the holomorphicity equations (1) for the O(n) loop model on the square lattice [11]. This is a dilute loop model, defined by the vertices in figure 4, where each closed loop carries a weight given by

$$n = -2\cos 2\eta, \qquad 0 \leqslant \eta \leqslant \frac{\pi}{2}.$$

Note that we have grouped the anisotropic weights so that there is symmetry under reflections in the diagonal axes. This symmetry is preserved when the plaquettes are deformed into rhombi. Since every loop configuration has an even number of plaquettes of type u_1 or u_2, the change $(u_1, u_2) \to (-u_1, -u_2)$ does not affect the Boltzmann weights.

We consider the observable $F_s(z)$, defined similarly to (6), except that the sum is now over graphs where there is an *oriented open path* going from 0 to z. A similar observable

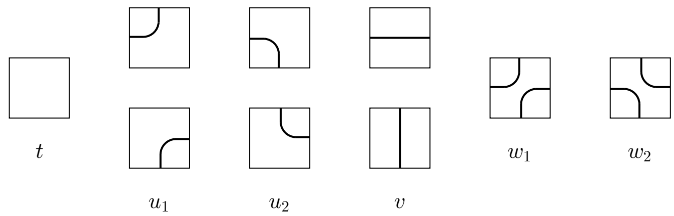

Figure 4. Vertices of the O(n) loop model on the square lattice. Each closed loop has a weight n.

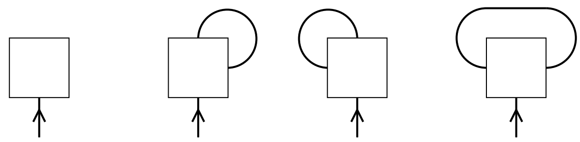

Figure 5. Loop configurations with one edge of the plaquette $\mathcal{P}$ connected to point 0.

has been considered by Smirnov [13] for the case of the hexagonal lattice. Now consider the contributions to the holomorphicity equation (1) where the first time the oriented curve, starting at 0, enters the chosen plaquette $\mathcal{P}$ is through a particular edge, for example the lowermost in figure 5. There are four inequivalent external connectivities as shown. We imagine these to be fixed, summing over all internal configurations consistent with them. This yields the linear system for the Boltzmann weights:

$$t + \mu u_1 - \mu\lambda^{-1}u_2 - v = 0, \tag{15a}$$

$$-\lambda^{-1}u_1 + nu_2 + \lambda\mu v - \mu\lambda^{-1}(w_1 + nw_2) = 0, \tag{15b}$$

$$nu_1 - \lambda u_2 - \mu\lambda^{-2}v + \mu(nw_1 + w_2) = 0, \tag{15c}$$

$$-\mu\lambda^{-2}u_1 + \mu\lambda u_2 + nv - \lambda^{-2}w_1 - \lambda^2 w_2 = 0, \tag{15d}$$

where we have set $\lambda = \mathrm{e}^{\mathrm{i}\pi s}$, $\varphi = (s+1)\alpha$, $\mu = \mathrm{e}^{\mathrm{i}\varphi}$. For real Boltzmann weights, (15a)–(15d) are four complex linear equations for six real unknowns $(t, u_1, u_2, v, w_1, w_2)$, and we have the relations

$$\mathrm{Im}[(n+1)(15a) - \lambda\mu^{-1}(15b) + \mu^{-1}(15c)] = 0,$$
$$\mathrm{Im}[\lambda\mu^{-1}(\lambda^2 - n\lambda^{-2})(15b) + \mu^{-1}(n\lambda^2 - \lambda^{-2})(15c) - (n^2-1)(15d)] = 0.$$

Thus, we can generally reduce (15a)–(15d) to a 6×6 real system.

There are two classes of solutions, for vanishing and non-vanishing v. First, if $v = 0$, then the configurations corresponding to (15d) never occur, and so this equation does not hold. In the special case $n = 1$, there exists a non-trivial solution *for any value of* s:

$$t = \sin\pi s, u_1 = \sin(\varphi - \pi s), \qquad u_2 = \sin\varphi, \qquad w_1 + w_2 = \sin\pi s. \tag{16}$$

This model can be mapped onto the six-vertex model (see figure 6), with weights $\omega_1 = \omega_2 = \sin(\varphi - \pi s)$, $\omega_3 = \omega_4 = \sin\varphi$, $\omega_5 = \omega_6 = \sin\pi s$. The corresponding anisotropy parameter [4] is $\Delta = \cos\pi s$. This is an example of a model admitting a holomorphic observable on the lattice, but for which the scaling limit of the corresponding curve cannot be described by simple SLE. This is because, for ordinary SLE, the central charge of the CFT is directly related

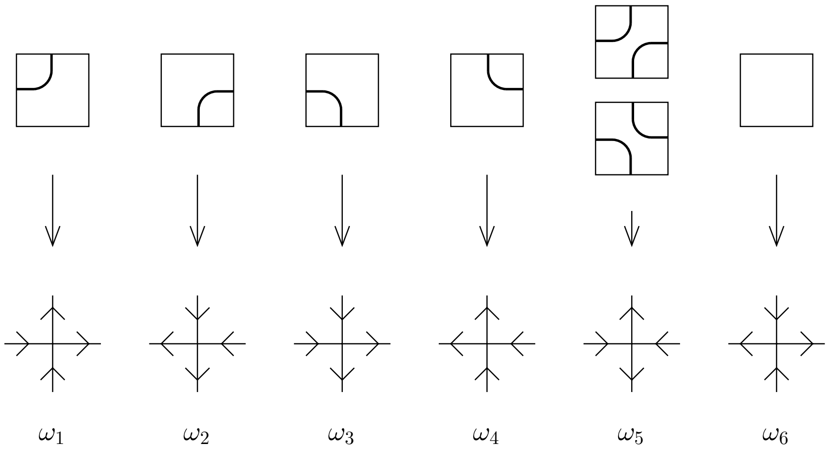

Figure 6. Mapping of the O(n) model onto the six-vertex model for $n = 1$, $v = 0$.

to the SLE parameter κ [8, 9] and hence to the conformal spin s: $c = 2s(5 - 8s)/(2s + 1)$. In the present case, since the boundary conditions for the six-vertex model are not twisted, its scaling limit has central charge $c = 1$ for all Δ. However the conformal spin s varies continuously with Δ. Therefore, the scaling limit of the curve can be SLE, with $\kappa = 4$, for at most one value (in fact $\Delta = 1/\sqrt{2}$). We conjecture that other values of Δ correspond to SLE $(4, \rho)$.

For $v = 0$, $n \neq -1$, we get a 5×5 linear system, with determinant $(n^2 - 1)^2 \sin\varphi \sin(\varphi - \pi s)$. Imposing $\sin\varphi = 0$ yields $(s + 1)\alpha = m\pi$ and in turn $s = m'$, where m, m' are integers. Thus, for the solution to exist at any value of α, we have to set $s = -1$. The Boltzmann weights are then

$$t = -u_1 - u_2, \qquad w_1 = -u_1, \qquad w_2 = -u_2. \tag{17}$$

The solution of the case $\sin(\varphi - \pi s) = 0$ is similar, and leads to the same Boltzmann weights and spin $s = -1$. If we change the sign of u_1, u_2, then the model (17) is equivalent to the dense loop model of section 2, with parameters $\sqrt{Q} = n + 1$, $a = u_1$, $b = u_2$. To see this, fill empty spaces with loops of weight 1 (*ghost loops*). The local weights do not depend on the type of loops involved (actual or ghost loops), so each loop has an overall weight $n + 1$. As a consequence, the dense loop model has a lattice antiholomorphic observable ($s < 0$), besides the holomorphic one found in [5]. However, several arguments rule out the hypothesis that $F_{s=-1}(z)$ corresponds to an antiholomorphic field in the continuum limit. First, $F_{s=-1}(z)$ is lattice antiholomorphic for any $Q > 0$, whereas it is well known that the self-dual Potts model is only critical for $0 \leqslant Q \leqslant 4$. Furthermore, the ratio u_1/u_2 in (17) does not depend on the angle α, which means that the same model has an antiholomorphic observable for any deformation angle: this is not acceptable physically in the continuum limit. So we conclude that, in the case $v = 0$ and generic $n \neq -1$, the holomorphicity conditions (1) for the dilute O(n) model merely lead to the case of the dense loop model, but the corresponding $F_s(z)$ is not a candidate for an antiholomorphic field in the continuum limit.

Let us now discuss the solutions of second class ($v \neq 0$), for a generic value of n. We get the 6×6 real system:

$$\{\mathrm{Re}(15a), \mathrm{Re}(15b), \mathrm{Im}(15b), \mathrm{Re}(15c), \mathrm{Im}(15c), \mathrm{Re}(15d)\},$$

with determinant: $(n^2-1)\sin\varphi\sin(\varphi-\pi s)(2\cos 4\pi s-3n+n^3)$. Non-trival solutions exist if the spin satisfies

$$\cos 4\pi s = \cos 6\eta. \tag{18}$$

The various solutions to (18) can be parametrized by extending the range of η to $[-\pi,\pi]$, and setting

$$s = \frac{3\eta}{2\pi} - \frac{1}{2}. \tag{19}$$

Then, we get the second class of solutions, with Boltzmann weights

$$\begin{aligned}
t &= -\sin(2\varphi - 3\eta/2) + \sin 5\eta/2 - \sin 3\eta/2 + \sin\eta/2,\\
u_1 &= -2\sin\eta\cos(3\eta/2-\varphi),\\
u_2 &= -2\sin\eta\sin\varphi,\\
v &= -2\sin\varphi\cos(3\eta/2-\varphi),\\
w_1 &= -2\sin(\varphi-\eta)\cos(3\eta/2-\varphi),\\
w_2 &= 2\cos(\eta/2-\varphi)\sin\varphi.
\end{aligned} \tag{20}$$

A remarkable fact is that the weights (20) are a solution of the Yang–Baxter equations for the O(n) loop model on the square lattice. Indeed, after a change of variables $\varphi \to \psi + (\pi+\eta)/4$, (20) coincides with the integrable weights in [11]. So, by solving the holomorphicity equations (15*a*)–(15*d*) on a deformed lattice, we recovered the integrable weights.

Now we interpret our findings in terms of CFT. First, in the regime $0 \leqslant \eta \leqslant \pi$ (branches 1 and 2 in [16]), on the basis of numerical diagonalization of the transfer matrix [16] and analysis of the Bethe ansatz equations [17, 18], it has been argued that the model (20) has the same continuum limit as the O(n) model on the hexagonal lattice [19]: it is described by a Coulomb gas of coupling constant $g = 2\eta/\pi$, with central charge and critical exponents given by (14). The value of the spin can then be written as $s = h_{2,1}$. In the regime $-\pi \leqslant \eta \leqslant 0$, the continuum limit is no longer described by a simple Coulomb gas. Rather, it has central charge

$$c = \frac{3}{2} - \frac{6(1-g')^2}{g'}, \tag{21}$$

where $g' = 2(\pi+\eta)/\pi$. The relation between s (19) and c (21), which is $c = (2-5s-16s^2)/(2s+2)$, is not consistent with SLE, so this is another example of a model with a holomorphic observable whose curves are not described by simple SLE. Note that in this case c may be greater than one, so some variant of SLE such as that considered by Bettelheim *et al* [20] may be the correct description.

4. The $C_2^{(1)}$ loop model

A simple generalization of the loop model of section 2 to a model with two loop colours was introduced in [12]. It is a dense loop model on the square lattice, where each loop can be either black or grey, and carries a weight n given by

$$n = -2\cos 2\eta, \qquad 0 \leqslant \eta \leqslant \frac{\pi}{2}.$$

The vertices of the model are shown in figure 7. For fixed n, the model has a one-parameter family of Boltzmann weights that satisfy the Yang–Baxter equations [12].

To describe the observable $F_s(z)$ that will be shown to be holomorphic, we first introduce a special kind of defect in the loop model: this defect sits on the midpoint of a plaquette edge,

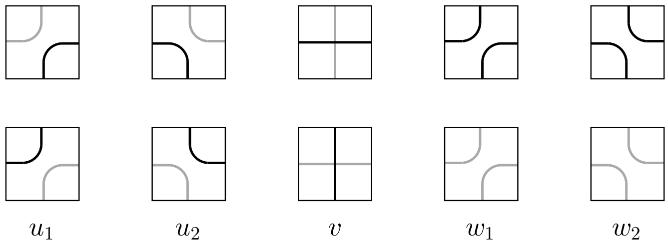

Figure 7. Vertices of the $C_2^{(1)}$ loop model on the square lattice. Each closed loop has a weight n.

and consists in a change of colour for the loop passing through. The observable $F_s(z)$ is then defined similarly to (6), except that the sum is now over graphs G where 0 and z belong to the same loop, and each of these points carries a defect. The angle $\theta(z)$ is defined as the sum of the windings of each loop strand from 0 to z. For a given plaquette $\mathcal{P}$ in (1), the graphs that contribute are those for which two edges of the plaquette $\mathcal{P}$ are connected to the point 0, and one edge contains a defect. The holomorphicity conditions (1) yield the system

$$nu_1 - \lambda u_2 - \mu\lambda^{-1}v + \mu(nw_1 + w_2) = 0, \tag{22a}$$

$$-\lambda^{-1}u_1 + nu_2 + \mu v - \mu\lambda^{-1}(w_1 + nw_2) = 0, \tag{22b}$$

$$-\mu\lambda^{-1}u_1 + \mu u_2 + nv - \lambda^{-1}w_1 - \lambda w_2 = 0, \tag{22c}$$

where we have set $\lambda = e^{2i\pi s}$, $\varphi = (2s+1)\alpha$, $\mu = e^{i\varphi}$. For real Boltzmann weights, we have the linear relation

$$\mathrm{Im}[\mu^{-1}(n\lambda - \lambda^{-1})(22a) + \lambda\mu^{-1}(\lambda - n\lambda^{-1})(22b) + (n^2-1)(22c)] = 0.$$

Thus, for $n^2 \neq 1$, we can ignore Im (22c), and the remaining 5×5 real system has determinant $(n^2-1)\sin\varphi\sin(\varphi - 2\pi s)\left(2\cos 4\pi s - 3n + n^3\right)$. It has a non-trivial solution for

$$\cos 4\pi s = \cos 6\eta. \tag{23}$$

There are two branches of solutions, which can be obtained by setting $s = (3\eta - \pi)/(2\pi)$, and extending the range of η to $[0, \pi]$. The Boltzmann weights then read

$$\begin{aligned} u_1 &= \sin\eta\sin(\varphi - 3\eta), \\ u_2 &= -\sin\eta\sin\varphi, \\ v &= -\sin\varphi\sin(\varphi - 3\eta), \\ w_1 &= -\sin(\varphi-\eta)\sin(\varphi - 3\eta), \\ w_2 &= -\sin(\varphi - 2\eta)\sin\varphi. \end{aligned} \tag{24}$$

These are exactly the integrable weights given in [12], with the change of variables $(\eta \to \lambda, \varphi \to u)$. So the $C_2^{(1)}$ loop model is another example where the solution of the lattice holomorphicity equations (1) satisfies also the Yang–Baxter equations.

5. Summary

In this communication we have identified several more examples of discretely holomorphic parafermionic observables in lattice models, and shown that this requirement always appears to pick out the critical points which are also integrable in the sense of Yang–Baxter. It would, of course, be important to have a more general understanding of this. It may give a simpler route to finding new integrable models. These parafermions are natural discrete candidates for parafermionic holomorphic conformal fields in the scaling limit, and thus can also suggest new structures, in particular, CFTs. In some cases the models we have considered correspond to CFTs with $c \geqslant 1$, so the scaling limit of the corresponding curves is probably described by some modification of simple SLE. The loop models we have considered may also be mapped to generalized restricted solid-on-solid (RSOS) models, and, from this point of view, it would be interesting to understand the coset construction of the corresponding CFT and its relation to the parafermionic fields.

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